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COMPTON SCATTERING BY STATIC AND MOVING MEDIA. I. THE TRANSFER EQUATION AND ITS MOMENTS

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ABSTRACT

Compton scattering of photons by nonrelativistic particles is thought to play an important role in forming the radiation spectrum of many astrophysical systems. Here we derive the time-dependent photon kinetic equation that describes spontaneous and induced Compton scattering, as well as absorption and emission by static and moving media, the corresponding radiative transfer equation, and their zeroth and first angular moments, both in the system frame and in the frame comoving with the medium. We show that it is necessary to use the correct relativistic differential scattering cross section in order to obtain a photon kinetic equation that is correct to first order in ε/m_e, T_e/m_e, and V, where ε is the photon energy, T_e and m_e are the electron temperature and rest mass, and V is the electron bulk velocity in units of the speed of light. We also demonstrate that the terms in the radiative transfer equation that are second order in V should usually be retained, because if the radiation energy density is sufficiently large, compared to the radiation flux, the effects of bulk Comptonization described by the terms that are second order in V can be as important as the effects described by the terms that are first order in V, even when V is small. The system- and fluid-frame equations that we derive are correct to first order in ε/m_e. Our system-frame equations, which are correct to second order in V, may be used when V is not too large. Our fluid-frame equations, which are exact in V, may be used when $V \rightarrow 1$. Both sets of equations are valid for systems of arbitrary optical depth and can therefore be used in both the free-streaming and diffusion regimes. We demonstrate that Comptonization by the electron bulk motion occurs whether or not the radiation field is isotropic or the bulk flow converges and that it is more important than thermal Comptonization if $V^2 > 3T_e/m_e$.

Subject headings: plasmas — radiation mechanisms: thermal — radiative transfer — scattering

1. INTRODUCTION

Compton scattering of photons by nonrelativistic particles is thought to play an important role in many astrophysical settings, including the early universe (see Peebles 1971; Sunyaev & Zeldovich 1980), clusters of galaxies (see Rephaeli 1995), active galactic nuclei (see Mushotzky, Done, & Pounds 1993), compact galactic X-ray and γ-ray sources (see Pozdnyakov, Sobol, & Sunyaev 1983), and supernova remnants (see McCray 1993). In this process, photons lose energy to the electrons or gain energy from their thermal and bulk motions, as a result of Compton recoil.

Starting from the Boltzmann equation for photons, Kompaneets (1957) derived the so-called Kompaneets equation, which describes the time evolution of the photon energy distribution caused by scattering by thermal electrons when there is no bulk motion, the radiation field is perfectly isotropic, and the change in the energy of the photon in each scattering is small. The conditions assumed in deriving the Kompaneets equation are never strictly satisfied in astrophysical systems, since the radiation field is always anisotropic near their boundaries. In inhomogeneous systems, the radiation field may be anisotropic even in the interior. Also, in many astrophysical systems the scattering particles have substantial bulk motions. Thus, although the Kompaneets equation has been used extensively to treat astrophysical systems, in many cases it does not give accurate results. This has motivated several authors to derive photon kinetic equations that are valid under more general conditions.

Comptonization in regions where the radiation field is anisotropic has been studied either by using Monte Carlo techniques or by solving partial differential equations for the specific intensity of the radiation field using finite-difference methods. Pozdnyakov et al. (1983) have reviewed Monte Carlo methods and results. Nagirner & Poutanen (1994) have reviewed work based on calculation of the complete Compton scattering kernel for polarized radiation. Babuel-Peyrissac & Rouvillois (1969), Pomraning (1973), Payne (1980), Madej (1989, 1991), and Titarchuk (1994) derived radiative transfer equations that describe Comptonization of an anisotropic radiation field when there is no bulk motion. Chan & Jones (1975), Blandford & Payne (1981a), and Fukue, Kato, & Matsumoto (1985) derived photon kinetic equations for thermal particles with nonzero bulk velocity in the diffusion approximation. Thorne (1981; see also Thorne, Flammang, & Żytkow 1981) derived fluid-frame moments of the radiative transfer equation in general relativity. The equations derived by these various sets of authors have been widely used to study Comptonization by strong shocks and accretion flows onto compact objects (see, e.g., Blandford & Payne 1981b; Payne & Blandford 1981; Lyubarskij & Sunyaev 1982; Colpi 1988; Riffert 1988; Mastichiadis & Kylafis 1992; Miller & Lamb 1992; Titarchuk & Lyubarskij 1995; Turolla et al. 1996; Titarchuk, Mastichiadis, & Kylafis 1997).

In the course of our investigation of the effects of Comptonization on the X-ray spectra of accreting neutron stars (see, e.g., Lamb 1989; Miller & Lamb 1992; Psaltis, Lamb, & Miller 1995) we have rederived the radiative transfer equation and its moments for static and moving media and found important corrections to almost all the above derivations of the photon kinetic or radiative transfer equations, as we explain in § 2. There we show that it is necessary to use the correct relativistic

differential scattering cross section in order to obtain a photon kinetic equation that is correct to first order in ε/m_e, T_e/m_e, and V, where ε is the photon energy, T_e and m_e are the electron temperature and rest mass, and V is the electron bulk velocity in units of the speed of light (we use units in which the Boltzmann constant and the speed of light are equal to unity). In § 2 we also demonstrate that the terms in the radiative transfer equation that are second order in V usually should be retained, because in many situations the second-order terms can be as important as the first-order terms, even when V is small (see also Yin & Miller 1995). If the terms that are second order in V are instead neglected, significant errors are introduced in the photon kinetic equation and its moments.

In § 3 we state our assumptions and approximations and introduce our notation. In § 4 we derive the time-dependent photon kinetic equation that describes spontaneous and induced scattering by static and moving media, the corresponding radiative transfer equation, and their zeroth and first angular moments, in the system frame and in the frame comoving with the medium. We derive the moment equations as well as the kinetic and transfer equations, because, although it is usually necessary to solve the full radiative transfer equation in order to determine accurately the angular distribution of the radiation field, the moment equations can be used to speed up the numerical computation by a large factor (Mihalas 1980; see also Mihalas 1978, p. 157) and to provide additional physical insight. The system- and fluid-frame equations that we derive are correct to first order in ε/m_e. Our system-frame equations, which are correct to second order in V, may be used when V is not too large. Our fluid-frame equations, which are exact in V, may be used when $V \to 1$. Both sets of equations are valid for systems of arbitrary optical depth and can therefore be used in both the free-streaming and diffusion regimes. Our equations can easily be generalized to describe scattering by an arbitrary number of particle species. Finally, in § 5 we summarize our results and their implications for Comptonization by static and moving media.

In Appendix A we give the photon kinetic and radiative transfer equations that are obtained by averaging the equations for a single electron over a drifting, relativistic Maxwellian electron velocity distribution. In Appendix B we give the radiative transfer equation that describes absorption and emission in moving media and its zeroth and first moments. There we point out that the addition of a photon source term in the transfer equation without any corresponding absorption term (see, e.g., Blandford & Payne 1981a) is fundamentally inconsistent with thermodynamics and leads to a radiative transfer equation that has a different mathematical character from that of the thermodynamically consistent equation.

2. MOTIVATION

The radiative transfer equation that describes scattering of photons by particles is an integrodifferential equation in which only derivatives with respect to the spatial coordinates appear (see, e.g., Nagirner & Poutanen 1994). The scattering kernel in this equation is nonlocal in photon energy and depends on the (possibly complicated) correlations between the angular dependence of the specific intensity of the radiation field, the velocity distribution of the particles, and the differential scattering cross section. In order to accelerate numerical calculations, gain better physical insight, and facilitate comparison with previous studies that made similar approximations, we convert this integrodifferential equation into a partial differential equation over the spatial coordinates and photon energy by expanding the scattering kernel in powers of the dimensionless quantities ε/m_e, T_e/m_e, and V, which we assume are small compared to unity. In this way, the scattering kernel in the transfer equation becomes local in photon energy, and the scattering integral over solid angle can be expressed in terms of the angular moments of the specific intensity of the radiation field.

We are primarily interested in deriving a transfer equation that can be used to calculate the spectra of X-ray and soft γ-ray sources; therefore in expanding the scattering kernel, we shall keep only terms of zeroth and first order in $\Delta\varepsilon_0/\varepsilon_0 \approx \varepsilon/m_e \ll 1$, which is the average fractional decrease in the energy of a photon in a single scattering in the electron rest frame (see eq. [17] below). Then, in order to obtain a radiative transfer equation of consistent accuracy, the terms in the expansion of the kernel in powers of T_e/m_e and V that are of the same size as the terms of first order in $\Delta\varepsilon_0/\varepsilon_0$, i.e., of the same size as ε/m_e, must be retained. Depending on the situation, terms of different order in T_e/m_e and V may need to be included.

We now discuss the accuracy of the expansion of the differential scattering cross section needed to obtain a transfer equation that is accurate to first order in ε/m_e, the orders of the terms in V that must be retained, and some subtle points that must be taken into account if the diffusion approximation is used.

2.1. Approximate Scattering Cross Section

In order to obtain a transfer equation that is consistently accurate to first order in ε/m_e, it is necessary to use the correct relativistic expression for the differential scattering cross section in the frame in which one is working. We shall work in the electron rest frame, and we therefore use the Klein-Nishina expression for the differential scattering cross section (see eq. [16]). To first order in ε/m_e, the Klein-Nishina cross section is

$$\frac{d\sigma}{d\Omega_0} \approx \frac{3\sigma_T}{16\pi} [1 + (\hat{l}_0 \cdot \hat{l}_0)^2]\left[1 - 2\frac{\varepsilon}{m_e}(1 - \hat{l}_0 \cdot \hat{l}_0)\right], \tag{1}$$

where σ_T is the total cross section for Thomson scattering and $\hat{l}_0$ and $\hat{l}_0$ are the direction vectors of the photon in the electron rest frame before and after the scattering.

Use of the Thomson approximation for the differential scattering cross section (Chan & Jones 1975; Payne 1980; Madej 1989, 1991) rather than equation (1) introduces errors in the radiative transfer and moment equations of order ε/m_e, i.e., of the same size as the basic Compton effect. Neglecting the angle dependence of the term of order ε/m_e in equation (1) (Pomraning 1973; Blandford & Payne 1981a) introduces errors in the radiative transfer equation of this same order, because, unlike the Thomson approximation, equation (1) is not forward-backward symmetric. Approximating equation (1) by the average of the cross section over the electron velocity distribution (Titarchuk 1994) neglects the effects of correlations between the angle dependence of the differential cross section, the specific intensity, and the electron velocity, introducing errors in the radiative

transfer equation of order ε/m_e, T_e/m_e, and V. Finally, use in the system frame of the Thomson or Klein-Nishina expressions for the differential scattering cross section (Pomraning 1973; Payne 1980; Madej 1989, 1991) introduces errors in the transfer equation of order V and T_e/m_e, because these expressions are valid only in the electron rest frame.

2.2. Importance of the Terms of Order V^2

It has long been recognized that if the divergence of the electron bulk velocity is nonzero, the electron bulk motion causes a secular change in the energy of the photons (see, e.g., Chan & Jones 1975; Blandford & Payne 1981a; Fukue et al. 1985). The photon kinetic equations derived in these works do not include any terms of second order in V and predict that photons are systematically upscattered in energy by the electron bulk motion if the flow is converging. However, as we show with the examples that follow, photons are systematically upscattered by the electron bulk motion even if the terms that are first order in V have, on average, no effect. Indeed, in some circumstances photons are systematically upscattered by the bulk motion even if the flow is *diverging*.

To see this, consider a situation in which the bulk motion of the electrons can be described as isotropic turbulence in which the velocity correlation length is much smaller than the photon mean free path and the bulk velocity satisfies $\mathbf{V} \cdot \mathbf{V} = 0$. If all the terms that are first order in V are included correctly in the photon kinetic equation, but terms that are second order in V are not included (Fukue et al. 1985), the zeroth and first moments of the kinetic equation will indicate that the electron bulk motion has no effect on the photon energy distribution on timescales longer than the velocity correlation time. In reality, however, the effect of the electron turbulent motions under the stated conditions is completely analogous to the effect of electron thermal motions, that is, the turbulent motions cause the mean energy of the photons to increase steadily with time at a rate that is proportional to the second moment of the turbulent velocity field. Hence, the electron bulk motion causes the mean photon energy to increase with time, even though the flow is not converging; in fact, in this case no terms that are first order in V have any effect on the radiation field, on average. If the turbulent velocity is high enough, the terms that are second order in V will cause the mean photon energy to increase even in a diverging flow.

Origin of terms that are second order in V.—The origin of the terms in the photon kinetic equation that are second order in V can be understood by considering a cold ($T_e = 0$) electron fluid with a uniform bulk velocity V in the system frame. For simplicity, let us analyze the scattering of photons in the Thomson limit in the electron rest frame. In this limit, the angular distribution of the scattered photons is backward-forward symmetric in the electron rest frame, so the average photon energy in the system frame after scattering is $\gamma^2(1 - \langle \hat{k} \cdot V \rangle)$ times the energy before the scattering (see Rybicki & Lightman 1979, p. 198), where $\gamma \equiv (1 - V^2)^{-1/2}$ and the average is over the initial photon wavevectors k in the system frame. Consider first the case of a photon distribution that is isotropic in the system frame. In this case the average photon energy in the system frame after scattering is γ^2 times the energy before scattering, so in each scattering the average energy of a photon is increased by an amount of order V^2 and hence there must be terms of order V^2 in the kinetic equation that describe this effect. In the more general case of a photon distribution that is anisotropic in the system frame, these terms of order V^2 almost always produce a secular increase in the average energy of the photons, but there are also terms of order V in the kinetic equation that may cause the average energy of the photons to increase or decrease, depending on the velocity field and on the angular distribution of the photons (there are generally terms of order V even if $\mathbf{V} \cdot \mathbf{V} = 0$; see Fukue et al. 1985).

Relative sizes of terms of different order in V.—The order in V of a given term in the various moment equations does not by itself determine whether the term should be retained in solving a given transport problem. This is because the terms of different order in V involve different angular moments of the radiation field, so the relative sizes of the moments must also be considered in determining which terms should be retained (see also Yin & Miller 1995).

For example, in the zeroth moment of the radiative transfer equation, which is a scalar equation for the energy density of the radiation field, the terms that are first order in V involve the scalar product of the vector quantity V with the vector quantity defined by the radiation field, i.e., the radiation flux H (see eq. [34]). In contrast, the terms that are second order in V do not involve H but instead involve the scalar quantity defined by the radiation field, i.e., the radiation energy density J, and the radiation stress-energy tensor K^{ij} (again see eq. [34]). In general, $V_i V_j K^{ij}$ is of the same size as $V^2 J$. Hence, if J is sufficiently large compared to H, the terms in the zeroth moment of the radiative transfer equation that are second order in V can be as large as the terms that are first order in V.

In the first moment of the radiative transfer equation, which is a vector equation for the radiation flux, the situation is reversed in that the terms that are first order in V involve J and K^{ij} (see eq. [40]), whereas the terms that are second order in V involve H and Q^{ijk}. In general, $V_j V_k Q^{ijk}$ is of the same size as $V(V \cdot H)$. Hence, one might be tempted to neglect the terms of order V^2 in the first moment of the radiative transfer equation if J is large compared to H, even if the terms of order V^2 are retained in the zeroth moment equation. However, this is in general unsafe, because it involves treating differently the same terms in the radiative transfer equation, from which both moment equations are derived, in computing the zeroth and first moments. In addressing a given transport problem, one can only determine which terms in the expansion in powers of V must be kept by considering the boundary conditions as well as the transfer equation.

The radiation field quantities that appear in terms involving only *odd* powers of V are all of about the same size. Similarly, the radiation field quantities that appear in terms that involve only *even* powers of V are all of about the same size, although they may be much larger or smaller than the radiation field quantities that appear in the terms involving odd powers of V. Hence, when V is much less than 1, terms of order V^3 and higher may be safely neglected in the derivation of the transfer equation (see also Yin & Miller 1995).

2.3. Use of the Diffusion Approximation

In situations in which the change during a scattering of the energy of a photon as measured in the fluid frame can be neglected and the longest photon mean free path λ_{max} is much smaller than the smallest length scale L_{min} on which physical

variables change, the specific intensity in the frame comoving with the electron fluid $I_f(\varepsilon_f)$ is given to lowest order in λ_{max}/L_{min} by (see Mihalas & Mihalas 1984, p. 457)

$$I_f(\varepsilon_f) = J_f(\varepsilon_f) + 3\hat{l}_f \cdot H_f(\varepsilon_f) , \tag{2}$$

where $J_f(\varepsilon_f)$ and $H_f(\varepsilon_f)$ are the zeroth and first angular moments of $I_f(\varepsilon_f)$. The diffusion approximation (sometimes called the zeroth-order diffusion approximation; see Mihalas & Mihalas 1984, § 97) consists in *assuming* that $I_f(\varepsilon_f)$ is given *exactly* by equation (2). Then the first fluid-frame Eddington factor $f^{ij}(\varepsilon_f) \equiv K_f^{ij}(\varepsilon_f)/J_f(\varepsilon_f)$ is *exactly* equal to $\frac{1}{3}\delta^{ij}$; here $K_f^{ij}(\varepsilon_f)$ is the second moment of $I_f(\varepsilon_f)$. As is evident from equation (2), if the source function involves only the zeroth, first, and second moments of $I_f(\varepsilon_f)$, then in the diffusion approximation it is only necessary to solve the three equations consisting of the zeroth and first moments of the fluid-frame radiative transfer equation and the closure relation $f^{ij}(\varepsilon_f) = \frac{1}{3}\delta^{ij}$ for the three moments $J_f(\varepsilon_f)$, $H_f(\varepsilon_f)$, and $K_f^{ij}(\varepsilon_f)$. The specific intensity may then be calculated in this approximation using equation (2).

The moments of the specific intensity in the system frame do *not* satisfy equation (2), even when $\lambda_{max}/L_{min} \ll 1$. If equation (2) is mistakenly used to relate the specific intensity to its zeroth and first moments in the system frame (Blandford & Payne 1981a) rather than in the fluid frame, then errors of order V will be introduced in the radiative transfer equation and its moments, even if $\lambda_{max}/L_{min} \ll 1$ (see Fukue et al. 1985).

If one wants to use the diffusion approximation (2) to solve for the moments of the specific intensity *in the system frame*, one must solve the *four* equations consisting of the zeroth and first moments of the system-frame radiative transfer equation and the two closure relations involving the first two Eddington factors for the first four moments $J(\varepsilon)$, $H^i(\varepsilon)$, $K^{ij}(\varepsilon)$, and $Q^{ijk}(\varepsilon)$ of the specific intensity $I(\varepsilon)$ in the system frame. The reason is that when equation (2) is boosted into the system frame and the terms that are second order in V are retained, as in general they must be (see above), the third moment of the specific intensity $Q^{ijk}(\varepsilon)$ is introduced into the closure relation. (Fukue et al. 1985 were able to set up a closed system of equations consisting of the zeroth and first moments of the system-frame radiative transfer equation and a closure relation involving the first Eddington factor only because they neglected all the terms in the radiative transfer equation that are second order and higher in V.)

When, as in Compton scattering, the energy of a photon in the fluid frame changes in a scattering, equation (2) may not be accurate for all photon energies, even if the mean free path of a photon is independent of its energy and $\lambda_{max}/L_{min} \ll 1$. As an example, consider a slab that is infinite along the x- and y-axes but finite along the z-axis, in which the electrons are cold and have no bulk motion. Suppose the slab is illuminated from one side with monochromatic photons of energy ε_{in} propagating in the z-direction. Since the electrons are static, any photons that have scattered have lost energy to the electrons and hence no longer have energy ε_{in}. Therefore all photons anywhere in the slab with energies equal to ε_{in} have never been scattered and are still propagating in the z-direction. As a result, the angular distribution of these photons is not described accurately by equation (2), even if λ_{max}/L_{min} is very small. Indeed, *the Eddington factor at energy ε_{in} is equal to unity everywhere in the slab*. In this example, equation (2) is accurate only at energies sufficiently below the injection energy ε_{in}.

The electron bulk velocity V is a vector quantity, and hence the relative sizes of the terms of first and higher order in V in the expansion of the transfer equation and its moments depend strongly on the relation between the angular dependence of the bulk velocity, the radiation field, and the differential scattering cross section. Therefore, even when the diffusion approximation is only slightly inaccurate (as for example when the Eddington factor differs only slightly from $\frac{1}{3}$), this inaccuracy produces additional terms in the moment equations that are of second order in V and that are therefore of the same magnitude as the second-order terms that would be present if the radiation field were given exactly by equation (2).

For these reasons, we derive the radiative transfer and moment equations without making use of the diffusion approximation.

3. ASSUMPTIONS, DEFINITIONS, AND APPROXIMATIONS

In the sections that follow we assume that the electron gas is nondegenerate (electron occupation number $\ll 1$). For conciseness we shall refer to it as a fluid, without implying anything about whether it is collisional or collisionless. We assume that photons are scattered only by electrons, neglecting scattering by any other particles, and set $h = c = k_B = 1$, where h is Planck's constant, c is the speed of light, and k_B is Boltzmann's constant. We indicate quantities evaluated in the rest frame of a particular electron by a subscript zero and quantities evaluated in the inertial frame momentarily comoving (locally) with the fluid, which we call the "fluid frame," by a subscript f. The "system frame" may be any global inertial frame (such as the frame at rest with respect to the center of mass of the system, if it is inertial). Quantities evaluated in the system frame have no subscript.

Bulk velocity and temperature.—We define the fluid frame as the frame in which the energy and momentum flux density both vanish (Landau & Lifshitz 1987, p. 505). In this frame, collisions between the electrons tend to establish an isotropic velocity distribution.

The system-frame three-velocity u of a given electron is related to its fluid-frame three-velocity u_f by a Lorentz boost. The first and second moments of u in the system frame are

$$\langle u \rangle = V \tag{3}$$

and

$$\langle u^2 \rangle = \langle v^2 \rangle + V^2 . \tag{4}$$

Here V is the three-velocity of the fluid as measured in the system frame, the angle brackets denote the average over the velocity distribution of the electrons in the fluid, and

$$v \equiv u - V \tag{5}$$

is the peculiar velocity of an electron as measured in the system frame. If the electron momentum distribution in the fluid frame is a relativistic Maxwellian, then the second moment of the electron velocity distribution evaluated in the fluid frame is

$$\langle u_f^2 \rangle_f = \frac{3T_e}{m_e} + O\left(\frac{T_e^2}{m_e^2}\right),$$

(6)

and the second moment of the electron peculiar velocity distribution in the system frame is

$$\langle v^2 \rangle = \frac{3T_e}{m_e} + O\left(\frac{T_e}{m_e} V^2\right).$$

(7)

Description of the radiation field.—In § 4.1 and Appendix A we derive the equation that describes the evolution of a radiation field interacting with a moving electron fluid. We shall refer to this equation as the *photon kinetic equation* when it is written in terms of the photon mode occupation number and as the *radiative transfer equation* when it is written in terms of the specific intensity of the radiation field. The two descriptions are equivalent (see, e.g., Mihalas 1978, p. 32), but the radiative transfer equation is more often used in astrophysical problems.

In deriving the photon kinetic equation, we shall describe the radiation field by the number $n(\hat{l}, \varepsilon)$ of photons with energy ε propagating in direction $\hat{l}$ with a given polarization state (we suppress the dependence on polarization state, because we consider only unpolarized radiation). The first few moments of $n(\hat{l}, \varepsilon)$ are

$$n \equiv \frac{1}{4\pi} \int n(\hat{l}, \varepsilon) d\Omega,$$

(8)

$$n^i \equiv \frac{1}{4\pi} \int n(\hat{l}, \varepsilon) l^i d\Omega,$$

(9)

$$n^{ij} \equiv \frac{1}{4\pi} \int n(\hat{l}, \varepsilon) l^i l^j d\Omega,$$

(10)

$$n^{ijk} \equiv \frac{1}{4\pi} \int n(\hat{l}, \varepsilon) l^i l^j l^k d\Omega.$$

(11)

In the definitions (8)–(11), the dependence of the moments on position and photon energy have been suppressed for brevity. Here and below we display the dependence of the photon occupation number on $\hat{l}$ and ε in order to distinguish it clearly from its zeroth moment.

In writing the radiative transfer equation, we shall describe the radiation field by its specific intensity $I(\hat{l}, \varepsilon) \equiv 2\varepsilon^3 n(\hat{l}, \varepsilon)$; here the factor of 2 accounts for the two photon polarization states. The first few moments of $I(\hat{l}, \varepsilon)$ are

$$J \equiv \frac{1}{4\pi} \int I(\hat{l}, \varepsilon) d\Omega = 2\varepsilon^3 n,$$

(12)

$$H^i \equiv \frac{1}{4\pi} \int I(\hat{l}, \varepsilon) l^i d\Omega = 2\varepsilon^3 n^i,$$

(13)

$$K^{ij} \equiv \frac{1}{4\pi} \int I(\hat{l}, \varepsilon) l^i l^j d\Omega = 2\varepsilon^3 n^{ij},$$

(14)

$$Q^{ijk} \equiv \frac{1}{4\pi} \int I(\hat{l}, \varepsilon) l^i l^j l^k d\Omega = 2\varepsilon^3 n^{ijk},$$

(15)

where again we have suppressed the dependence of the moments on position and photon energy.

Compton scattering.—In the rest frame of the electron, the differential cross section for scattering of unpolarized radiation is (see Berestetskiĭ, Lifshitz, & Pitaevskiĭ 1971, p. 297)

$$\frac{d\sigma}{d\Omega_0} = \frac{3\sigma_T}{16\pi} \left(\frac{\varepsilon_0'}{\varepsilon_0}\right)^2 \left[\frac{\varepsilon_0}{\varepsilon_0'} + \frac{\varepsilon_0'}{\varepsilon_0} - 1 + (\hat{l}_0 \cdot \hat{l}_0')^2\right],$$

(16)

where σ_T is the Thomson cross section, ε_0 and $\hat{l}_0$ are the energy and direction of the incident photon, and

$$\varepsilon_0' = \frac{\varepsilon_0}{1 - (\varepsilon_0/m_e)(1 - \hat{l}_0 \cdot \hat{l}_0')}$$

(17)

and $\hat{l}_0'$ are the energy and direction of the scattered photon.

The energy and direction of propagation of a photon in the electron rest frame are related to the same quantities in the system frame by the Lorentz transformations

$$\varepsilon_0 = \gamma \varepsilon (1 - \hat{l} \cdot \boldsymbol{u}) = \frac{\varepsilon}{\gamma(1 + \hat{l}_0 \cdot \boldsymbol{u})} \tag{18}$$

and

$$\hat{l}_0 = \frac{\varepsilon}{\varepsilon_0} \left\{ \hat{l} + \left[\frac{\gamma - 1}{u^2} (\hat{l} \cdot \boldsymbol{u}) - \gamma \right] \boldsymbol{u} \right\} . \tag{19}$$

The photon phase-space volume is Lorentz invariant (see Mihalas 1978, p. 495), i.e.,

$$\varepsilon \, d\varepsilon \, d\Omega = \varepsilon_0 \, d\varepsilon_0 \, d\Omega_0 , \tag{20}$$

so

$$n(\hat{l}, \varepsilon) = n_0(\hat{l}_0, \varepsilon_0) . \tag{21}$$

Validity of the approximations.—In deriving the photon kinetic and radiative transfer equations and their moments in the system frame, we shall retain terms up to first order in ε/m_e and up to second order in u (first order in T_e/m_e), neglecting terms of order $(\varepsilon/m_e)u$ or higher. These equations are therefore valid when

$$\frac{\varepsilon}{m_e} \ll 1 \qquad \text{and} \qquad \frac{\varepsilon}{m_e} \left(V^2 + \frac{3T_e}{m_e} \right)^{1/2} \ll \left(V^2 + \frac{3T_e}{m_e} \right) \ll 1 . \tag{22}$$

These conditions are usually satisfied in accretion onto white dwarfs and neutron stars but are not satisfied in accretion onto black holes, because the flow velocity $V \to 1$ at the horizon.

Our expressions for the photon kinetic and radiative transfer equations in the fluid frame are correct to all orders in V but only to first order in T_e/m_e and ε_f/m_e. These equations are therefore valid when

$$\frac{\varepsilon_f}{m_e} \ll 1 \qquad \text{and} \qquad \frac{\varepsilon_f}{m_e} \left(\frac{3T_e}{m_e} \right)^{1/2} \ll \left(\frac{3T_e}{m_e} \right) \ll 1 \tag{23}$$

and can be used where $V \simeq 1$.

4. PHOTON KINETIC AND RADIATIVE TRANSFER EQUATIONS AND THEIR ZEROTH AND FIRST MOMENTS

4.1. *Photon Kinetic Equation*

The photon kinetic equation in the system frame is

$$k^\mu \partial_\mu n(k) = \int \frac{d^3k'}{(2\pi)^3} \int d^3p \{ W(k'p, kp')n(k')[1 + n(k)]f(p)[1 - f(p')] - W(kp, k'p')n(k)[1 + n(k')]f(p)[1 - f(p')] \} , \tag{24}$$

where $k^\mu = (\varepsilon, \varepsilon\hat{l})$, $n(k)$ is the photon occupation number, p is the electron momentum, $f(p)$ is the electron momentum distribution, $W(kp, k'p')$ is the transition rate for the scattering $k + p \to k' + p'$, and in writing the total derivative on the left side we have used the Einstein summation convention. The left side of equation (24) is manifestly covariant. The right side is also covariant if the appropriate transition rates are used.

It is convenient to integrate first over the photon states and then over the electron states, because the angular integrals are then much simpler. We will therefore assume for the moment that all the electrons have the same momentum p. Because each side of the photon kinetic equation (24) is covariant, the left side can be evaluated in the system frame and the right side in the electron rest frame at the wavevector k_0 corresponding to the wavevector k of the photon in the system frame. The resulting equation for electrons moving with velocity $\boldsymbol{u} = p/(\gamma m_e)$ is (cf. Peebles 1971, p. 204)

$$(\partial_t + l^i \partial_i)\varepsilon n(\hat{l}, \varepsilon) = \varepsilon_0 n_{e0} \int d\Omega_0 \left\{ \left(\frac{\varepsilon_0'}{\varepsilon_0} \right)^2 \frac{\partial \varepsilon_0'}{\partial \varepsilon_0} \left(\frac{d\sigma}{d\Omega'} \right)_0 n_0(\hat{l}_0', \varepsilon_0')[1 + n_0(\hat{l}_0, \varepsilon_0)] - \left(\frac{d\sigma}{d\Omega'} \right)_0 n_0(\hat{l}_0, \varepsilon_0)[1 + n_0(\hat{l}_0', \varepsilon_0')] \right\} , \tag{25}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x^i$, x^i are the spatial coordinates, n_{e0} is the electron density in the electron rest frame, $\hat{l}_0$ and ε_0 are related to $\hat{l}$ and ε by equations (19) and (18), and ε_0' is related to ε_0 by equation (17). The factor preceding the cross section in the first term of the collision integral is the Jacobian that corrects for the different phase spaces of the incident and scattered photons.

We evaluate the collision integral in equation (25) first by using the Lorentz invariance of the photon occupation number to relate $n_0(\hat{l}_0, \varepsilon_0')$ to $n(\hat{l}', \varepsilon')$ and then by relating $n(\hat{l}', \varepsilon')$ to $n(\hat{l}', \varepsilon)$ by expanding $n(\hat{l}', \varepsilon')$ to second order in u and to first order in ε/m_e. This gives (see Peebles 1971, p. 204)

$$n_0(\hat{l}_0, \varepsilon_0') \simeq n(\hat{l}', \varepsilon) + \left[(\hat{l}_0' - \hat{l}_0) \cdot \boldsymbol{u} + (\hat{l}_0 \cdot \boldsymbol{u})(\hat{l}_0' - \hat{l}_0) \cdot \boldsymbol{u} + \frac{\varepsilon}{m_e} (1 - \hat{l}_0 \cdot \hat{l}_0') \right] \varepsilon \partial_\varepsilon n(\hat{l}', \varepsilon) + \frac{1}{2} [(\hat{l}_0' - \hat{l}_0) \cdot \boldsymbol{u}]^2 \varepsilon^2 \partial_\varepsilon^2 n(\hat{l}', \varepsilon) , \tag{26}$$

where $\partial_\varepsilon \equiv \partial/\partial\varepsilon$ and $\partial_\varepsilon^2 \equiv \partial^2/\partial\varepsilon^2$. We then use this result, the Lorentz invariance of the photon distribution, and $(d\sigma/d\Omega')_0$ and $n_{e0}\varepsilon_0$, expanded to second order in u, to obtain the approximate photon kinetic equation that describes the effects of scattering by electrons with velocity u,

$$(\partial_t + l^i\partial_i)n(\hat{l}, \varepsilon) = \frac{3n_e\sigma_T}{16\pi} \int d\Omega_0'[1 + (\hat{l}_0 \cdot \hat{l}_0)^2]\left[1 - 2\frac{\varepsilon}{m_e}(1 - \hat{l}_0 \cdot \hat{l}_0)\right]\left\{[1 - (\hat{l}_0 \cdot u) + (\hat{l}_0 \cdot u)^2 - u^2]\right.$$

$$+ [(\hat{l}_0' - \hat{l}_0) \cdot u + 2(\hat{l}_0 \cdot u)(\hat{l}_0 - \hat{l}_0) \cdot u]\varepsilon\partial_\varepsilon + \frac{1}{2}[(\hat{l}_0' - \hat{l}_0) \cdot u]^2\varepsilon^2\partial_\varepsilon^2 + \frac{\varepsilon}{m_e}(1 - \hat{l}_0 \cdot \hat{l}_0)(4 + \varepsilon\partial_\varepsilon)$$

$$\left. + 2\frac{\varepsilon}{m_e}(1 - \hat{l}_0 \cdot \hat{l}_0)n(\hat{l}, \varepsilon)(2 + \varepsilon\partial_\varepsilon)\right\}n(\hat{l}', \varepsilon) - n_e\sigma_T\left(1 - 2\frac{\varepsilon}{m_e}\right)(1 - \hat{l} \cdot u)n(\hat{l}, \varepsilon) . \tag{27}$$

In Appendix A we give the photon kinetic and radiative transfer equations that are obtained by averaging equation (27) over a drifting, relativistic Maxwellian electron velocity distribution (see eqs. [3]–[7]). The moment equations derived in the next two subsections can be obtained by computing the zeroth and first moments of the equations given in Appendix A. Here we follow the simpler approach of first computing the moments of equation (27) and then averaging them over the electron velocity distribution.

4.2. Zeroth Moment and Radiation Energy Density

We compute the zeroth moment of the photon kinetic equation by first integrating both sides of equation (27) over all directions. Making use of the Jacobian

$$\frac{\partial(\Omega, \Omega_0')}{\partial(\Omega_0, \Omega')} = \left(\frac{\varepsilon_0}{\varepsilon}\frac{\partial\varepsilon_0}{\partial\varepsilon}\right)\left(\frac{\varepsilon'}{\varepsilon_0'}\frac{\partial\varepsilon'}{\partial\varepsilon_0'}\right) \simeq [1 - 2(\hat{l}_0 \cdot u) + 2(\hat{l}' \cdot u) - 2u^2 + 3(\hat{l}_0 \cdot u)^2 + 3(\hat{l}' \cdot u)^2 - 4(\hat{l}_0 \cdot u)(\hat{l}' \cdot u)] , \tag{28}$$

we find

$$\partial_t n + \partial_i n^i = \frac{3\sigma_T}{16\pi} n_e \int d\Omega' \int d\Omega_0[1 + (\hat{l}_0 \cdot \hat{l}_0)^2]\left[1 - \frac{2\varepsilon}{m_e}(1 - \hat{l}_0 \cdot \hat{l}_0)\right]$$

$$\times \left(\left\{[1 - 3u^2 + 3(\hat{l}' \cdot u)^2 + 2(\hat{l}' \cdot u)] + (\hat{l}_0 \cdot u)[1 + 2(\hat{l}' \cdot u)]\varepsilon\partial_\varepsilon + \frac{1}{2}(\hat{l}_0 \cdot u)^2\varepsilon^2\partial_\varepsilon^2 + \frac{\varepsilon}{m_e}(4 + \varepsilon\partial_\varepsilon)\right\}\right.$$

$$- (\hat{l}_0 \cdot u)\{[3 + 6(\hat{l}' \cdot u)] + [1 + 2(\hat{l}' \cdot u)]\varepsilon\partial_\varepsilon + (\hat{l}_0 \cdot u)\varepsilon^2\partial_\varepsilon^2\} + (\hat{l}_0 \cdot u)^2\left(6 + 4\varepsilon\partial_\varepsilon + \frac{1}{2}\varepsilon^2\partial_\varepsilon^2\right)$$

$$\left. - \frac{\varepsilon}{m_e}(\hat{l}_0 \cdot \hat{l}_0)(4 + \varepsilon\partial_\varepsilon) + \frac{2\varepsilon}{m_e}(1 - \hat{l} \cdot \hat{l}')n(\hat{l}, \varepsilon)(2 + \varepsilon\partial_\varepsilon)\right)n(\hat{l}', \varepsilon) - n_e\sigma_T\left[\left(1 - 2\frac{\varepsilon}{m_e}\right)n - n^iu_i\right] . \tag{29}$$

Next we integrate over $d\Omega_0$ and then over $d\Omega'$, using definitions (8)–(11). The result is

$$\frac{1}{n_e\sigma_T}(\partial_t n + \partial_i n^i) = (3 + \varepsilon\partial_\varepsilon)n^iu_i + \left[\frac{\varepsilon}{m_e}(4 + \varepsilon\partial_\varepsilon) + \frac{u^2}{3}(4\varepsilon\partial_\varepsilon + \varepsilon^2\partial_\varepsilon^2)\right]n + \left(\frac{36}{10} + \frac{34}{10}\varepsilon\partial_\varepsilon + \frac{11}{20}\varepsilon^2\partial_\varepsilon^2\right)\left(n^{ij}u_iu_j - \frac{1}{3}nu^2\right)$$

$$+ \frac{3}{2}\left(\frac{\varepsilon}{m_e}\right)(2n^2 - 2n^in^i + 2n^{ij}n^{ij} - 2n^{ijk}n^{ijk} + n\varepsilon\partial_\varepsilon n - n^i\varepsilon\partial_\varepsilon n^i + n^{ij}\varepsilon\partial_\varepsilon n^{ij} - n^{ijk}\varepsilon\partial_\varepsilon n^{ijk}) , \tag{30}$$

where we have again used the Einstein summation convention. Finally, after averaging equation (30) over the electron velocity distribution, we obtain the zeroth moment of the kinetic equation that describes the effects of scattering by a fluid of electrons, namely,

$$\frac{1}{n_e\sigma_T}(\partial_t n + \partial_i n^i) = (3 + \varepsilon\partial_\varepsilon)n^iV_i + \frac{\varepsilon}{m_e}(4 + \varepsilon\partial_\varepsilon)n + \frac{1}{3}(\langle v^2\rangle + V^2)(4\varepsilon\partial_\varepsilon + \varepsilon^2\partial_\varepsilon^2)n$$

$$+ \left(\frac{36}{10} + \frac{34}{10}\varepsilon\partial_\varepsilon + \frac{11}{20}\varepsilon^2\partial_\varepsilon^2\right)\left(n^{ij}\langle v_iv_j\rangle - \frac{1}{3}n\langle v^2\rangle\right) + \left(\frac{36}{10} + \frac{34}{10}\varepsilon\partial_\varepsilon + \frac{11}{20}\varepsilon^2\partial_\varepsilon^2\right)\left(n^{ij}V_iV_j - \frac{1}{3}nV^2\right)$$

$$+ \frac{3}{2}\left(\frac{\varepsilon}{m_e}\right)(2n^2 - 2n^in^i + 2n^{ij}n^{ij} - 2n^{ijk}n^{ijk} + n\varepsilon\partial_\varepsilon n - n^i\varepsilon\partial_\varepsilon n^i + n^{ij}\varepsilon\partial_\varepsilon n^{ij} - n^{ijk}\varepsilon\partial_\varepsilon n^{ijk}) . \tag{31}$$

This equation is valid for any arbitrary (possibly anisotropic) distribution of electron velocities in both the diffusion and free-streaming regimes.

If the electron velocity distribution in the fluid frame is a relativistic Maxwellian with temperature T_e, then (see eqs. [3]–[5] and [7])

$$\langle v_iv_j\rangle \simeq \frac{1}{3}\langle v^2\rangle\delta_{ij} \simeq \frac{T_e}{m_e}\delta_{ij} , \tag{32}$$

and the zeroth moment of the photon kinetic equation can be written, to the same accuracy as that of equation (31), as

$$\frac{1}{n_e\sigma_T}(\partial_t n + \partial_i n^i) = (3 + \varepsilon\partial_\varepsilon)n^i V_i + \left[\frac{\varepsilon}{m_e}(4 + \varepsilon\partial_\varepsilon) + \left(\frac{T_e}{m_e} + \frac{V^2}{3}\right)(4\varepsilon\partial_\varepsilon + \varepsilon^2\partial_\varepsilon^2)\right]n + \left(\frac{36}{10} + \frac{34}{10}\varepsilon\partial_\varepsilon + \frac{11}{20}\varepsilon^2\partial_\varepsilon^2\right)\left(n^{ij}V_i V_j - \frac{1}{3}nV^2\right)$$

$$+ \frac{3}{2}\left(\frac{\varepsilon}{m_e}\right)(2n^2 - 2n^i n^i + 2n^{ij}n^{ij} - 2n^{ijk}n^{ijk} + n\varepsilon\partial_\varepsilon n - n^i\varepsilon\partial_\varepsilon n^i + n^{ij}\varepsilon\partial_\varepsilon n^{ij} - n^{ijk}\varepsilon\partial_\varepsilon n^{ijk}),\quad(33)$$

where we have used the relation $n^{ij}\delta_{ij} = n$. The corresponding zeroth moment of the radiative transfer equation, with emission and absorption included, is (see Appendix B)

$$\frac{\partial J}{\partial t} + \frac{\partial}{\partial x^i}H^i = \left\{\varepsilon\partial_\varepsilon\left[\left(\frac{\varepsilon - 4T_e}{m_e}\right)J\right] + \frac{T_e}{m_e}\varepsilon\partial_\varepsilon^2(\varepsilon J) + \varepsilon\partial_\varepsilon H^i V_i + \frac{V^2}{3}\left[-4\varepsilon\partial_\varepsilon J + \varepsilon\partial_\varepsilon^2(\varepsilon J)\right]\right.$$

$$+ \left(\frac{1}{10}\varepsilon\partial_\varepsilon + \frac{11}{20}\varepsilon^2\partial_\varepsilon^2\right)\left(K^{ij}V_i V_j - \frac{1}{3}JV^2\right)$$

$$+ \frac{3}{4}\left(\frac{\varepsilon}{m_e}\right)\left[\left(\varepsilon\partial_\varepsilon J - J\right)\frac{J}{\varepsilon^3} - \left(\varepsilon\partial_\varepsilon H^i - H^i\right)\frac{H^i}{\varepsilon^3} + \left(\varepsilon\partial_\varepsilon K^{ij} - K^{ij}\right)\frac{K^{ij}}{\varepsilon^3} - \left(\varepsilon\partial_\varepsilon Q^{ijk} - Q^{ijk}\right)\frac{Q^{ijk}}{\varepsilon^3}\right]\right\}n_e\sigma_T$$

$$+ \left(1 - \frac{1}{6}V^2\varepsilon\partial_\varepsilon + \frac{1}{6}V^2\varepsilon^2\partial_\varepsilon^2\right)\eta_\varepsilon$$

$$- \left\{\left(J + \frac{1}{2}JV^2 - H^i V_i\right) + \left[K^{ij}V_i V_j + \frac{1}{2}JV^2 - H^i V_i\right]\varepsilon\partial_\varepsilon + \frac{1}{2}K^{ij}V_i V_j\varepsilon^2\partial_\varepsilon^2\right\}\chi_\varepsilon,\quad(34)$$

where η_ε and χ_ε are the emission and absorption coefficients, which are defined in the fluid frame but evaluated at the energy ε of the photons in the system frame.

4.3. First Moment and Radiation Flux

We compute the first moment of the photon kinetic equation by multiplying both sides of equation (27) by l and then integrating over all directions. In performing the integration we use the transformation (19) and the Jacobian (28), integrating first over $d\Omega_0$ and then over $d\Omega'$. The result is

$$\frac{1}{n_e\sigma_T}(\partial_t n^i + \partial_j n^{ij}) = -n^i - \frac{2}{5}\frac{\varepsilon}{m_e}(1 + \varepsilon\partial_\varepsilon)n^i + \frac{1}{5}(-nu_i + 3n^{ij}u_j) - \frac{1}{5}(8n^{ijk}u_j u_k - 2n^i u_j u_j) - \frac{1}{10}[3\varepsilon\partial_\varepsilon(nu_i) + \varepsilon\partial_\varepsilon(n^{ij}u_j)]$$

$$- \frac{1}{10}[10\varepsilon\partial_\varepsilon(n^j u_j)u_i + 9\varepsilon\partial_\varepsilon(n^{ijk}u_j u_k) - \varepsilon\partial_\varepsilon n^i u^2] - \frac{1}{10}[3\varepsilon^2\partial_\varepsilon^2(n^j u_j)u_i + \varepsilon^2\partial_\varepsilon^2(n^{ijk}u_j u_k)]$$

$$+ \frac{3}{2}\left(\frac{\varepsilon}{m_e}\right)(2nn^i - 2n^j n^{ij} + 2n^{jk}n^{ijk} - 2n^{jkl}n^{ijkl} + n^i\varepsilon\partial_\varepsilon n - n^{ij}\varepsilon\partial_\varepsilon n^j + n^{ijk}\varepsilon\partial_\varepsilon n^{jk} - n^{ijkl}\varepsilon\partial_\varepsilon n^{jkl}).\quad(35)$$

Averaging equation (35) over the electron velocity distribution, we finally obtain

$$\frac{1}{n_e\sigma_T}(\partial_t n^i + \partial_j n^{ij}) = -n^i - \frac{2}{5}\frac{\varepsilon}{m_e}(1 + \varepsilon\partial_\varepsilon)n^i + \frac{3}{5}\left(n^{ij}V_j - \frac{1}{3}nV_i\right) - \frac{1}{10}\varepsilon\partial_\varepsilon(n^{ij}V_j + 3nV_i)$$

$$- \frac{2}{5}(4n^{ijk}V_j V_k - n^i V_j V_j) - \frac{1}{10}\varepsilon\partial_\varepsilon(9n^{ijk}V_j V_k + 10n^j V_j V_i - n^i V_j V_j)$$

$$- \frac{1}{10}\varepsilon^2\partial_\varepsilon^2(n^{ijk}V_j V_k + 3n^j V_j V_i) - \frac{2}{5}(4n^{ijk}\langle v_j v_k\rangle - n^i\langle v_j v_j\rangle)$$

$$- \frac{1}{10}\varepsilon\partial_\varepsilon(n^{ijk}\langle v_j v_k\rangle + 10n^j\langle v_j v_i\rangle - n^i\langle v_j v_j\rangle) - \frac{1}{10}\varepsilon^2\partial_\varepsilon^2(n^{ijk}\langle v_j v_k\rangle + 3n^j\langle v_j v_i\rangle)$$

$$+ \frac{3}{2}\left(\frac{\varepsilon}{m_e}\right)(2nn^i - 2n^j n^{ij} + 2n^{jk}n^{ijk} - 2n^{jkl}n^{ijkl} + n^i\varepsilon\partial_\varepsilon n - n^{ij}\varepsilon\partial_\varepsilon n^j + n^{ijk}\varepsilon\partial_\varepsilon n^{jk} - n^{ijkl}\varepsilon\partial_\varepsilon n^{jkl}).\quad(36)$$

If the electron velocity distribution in the fluid frame is a relativistic Maxwellian with temperature T_e, then

$$n^j\langle v_j v_i\rangle = \frac{T_e}{m_e}n^i\quad(37)$$

and

$$n^{ijk}\langle v_j v_k\rangle = \frac{T_e}{m_e}n^i,\quad(38)$$

and the first moment of the photon kinetic equation reduces to

$$\frac{1}{n_e \sigma_T}(\partial_t n^i + \partial_j n^{ij}) = -\left[1 + \frac{2}{5}\frac{\varepsilon}{m_e}\left(1 + \partial_\varepsilon\right) + \frac{2}{5}\frac{T_e}{m_e}(1 + 4\varepsilon\partial_\varepsilon + \varepsilon^2\partial_\varepsilon^2)\right]n^i$$

$$+ \frac{3}{5}\left(n^{ij}V_j - \frac{1}{3}nV_i\right) - \frac{1}{10}\varepsilon\partial_\varepsilon(n^{ij}V_j + 3nV_i) - \frac{2}{5}(4n^{ijk}V_j V_k - n^i V_j V_j)$$

$$- \frac{1}{10}\varepsilon\partial_\varepsilon(9n^{ijk}V_j V_k + 10n^j V_j V_i - n^i V_j V_j) - \frac{1}{10}\varepsilon^2\partial_\varepsilon^2(n^{ijk}V_j V_k + 3n^j V_j V_i)$$

$$+ \frac{3}{2}\left(\frac{\varepsilon}{m_e}\right)(2nn^i - 2n^j n^{ij} + 2n^{jk}n^{ijk} - 2n^{jkl}n^{ijkl} + n^i\varepsilon\partial_\varepsilon n - n^{ij}\varepsilon\partial_\varepsilon n^j + n^{ijk}\varepsilon\partial_\varepsilon n^{jk} - n^{ijkl}\varepsilon\partial_\varepsilon n^{jkl}) . \quad (39)$$

The corresponding first moment of the radiative transfer equation with absorption and emission included is (again see Appendix B)

$$\partial_t H^i + \partial_j K^{ij} = \left\{-H^i - \frac{2}{5}\left(\frac{T_e - 3\varepsilon}{m_e}\right)H^i - \frac{2}{5}\left[\varepsilon\partial_\varepsilon\left(\frac{\varepsilon - 4T_e}{m_e}\right) + \frac{T_e}{m_e}\varepsilon\partial_\varepsilon^2\varepsilon\right]H^i + \frac{1}{10}\left[(9 - \varepsilon\partial_\varepsilon)K^{ij}V_j + (7 - 3\varepsilon\partial_\varepsilon)JV_i\right]\right.$$

$$+ \frac{1}{10}(-6 + 8\varepsilon\partial_\varepsilon - \varepsilon^2\partial_\varepsilon^2)H^j V_j V_i + \frac{1}{10}(1 + \varepsilon\partial_\varepsilon)H^i V_j V_j - \frac{1}{10}(1 + 3\varepsilon\partial_\varepsilon + \varepsilon^2\partial_\varepsilon^2)Q^{ijk}V_j V_k$$

$$+ \frac{3}{4}\left(\frac{\varepsilon}{m_e}\right)\left[(\varepsilon\partial_\varepsilon J - J)\frac{H^i}{\varepsilon^3} - (\varepsilon\partial_\varepsilon H^j - H^j)\frac{K^{ij}}{\varepsilon^3} + (\varepsilon\partial_\varepsilon K^{jk} - K^{jk})\frac{Q^{ijk}}{\varepsilon^3} - (\varepsilon\partial_\varepsilon Q^{jkl} - Q^{jkl})\frac{R^{ijkl}}{\varepsilon^3}\right]\right\}n_e\sigma_T$$

$$-\left[\left(1 + \frac{1}{2}V_j V_j\right)H^i - K^{ij}V_j + \left(\frac{1}{2}H^i V_j V_j - K^{ij}V_j + \frac{1}{2}Q^{ijk}V_j V_k\right)\varepsilon\partial_\varepsilon + \frac{1}{2}Q^{ijk}V_j V_k\varepsilon^2\partial_\varepsilon^2\right]\chi_\varepsilon + \frac{1}{3}V_i(2 - \varepsilon\partial_\varepsilon)\eta_\varepsilon ,$$

$$(40)$$

where R^{ijkl} is the fourth moment of the specific intensity.

4.4. Equations in the Fluid Frame

The photon kinetic and radiative transfer equations as well as their moments take their simplest form in the fluid frame, since $V = 0$ in this frame, by definition. These equations can be written choosing as independent variables either Eulerian coordinates fixed in space and time or Lagrangian coordinates comoving with the fluid.

In terms of the fluid-frame coordinates, the transfer equation (A8) and its zeroth and first moments, equations (34) and (40), become, in the fluid frame,

$$(\partial_{t_f} + l_f^i \partial_{i_f})I_f(\hat{l}_f, \varepsilon_f) = n_{e,f}\sigma_T\left[\mathscr{L}_1 I_f(\hat{l}_f, \varepsilon_f) + \frac{3}{4}(\mathscr{L}_2 J_f + \mathscr{L}_3^i H_f^i + \mathscr{L}_4^{ij}K_f^{ij} + \mathscr{L}_5^{ijk}Q_f^{ijk} + \mathscr{L}_6^{ijkl}R_f^{ijkl})\right.$$

$$\left. + \frac{3}{4\varepsilon^3}\left(\frac{\varepsilon_f}{m_e}\right)I_f(\hat{l}_f,\varepsilon_f)(\varepsilon_f\partial_{\varepsilon_f} - 1)(J_f - l_f^i H_f^i + l_f^i l_f^j K^{ij} - l_f^i l_f^j l_f^k Q^{ijk})\right]. \quad (41)$$

$$\partial_{t_f}J_f + \partial_{i_f}H_f^i = n_{e,f}\sigma_T\left\{\left[\varepsilon_f\partial_{\varepsilon_f}\left(\frac{\varepsilon_f - 4T_e}{m_e}\right) + \frac{T_e}{m_e}\varepsilon_f\partial_{\varepsilon_f}^2\varepsilon_f\right]J_f\right.$$

$$\left. + \frac{3}{4}\left(\frac{\varepsilon_f}{m_e}\right)\left[\left(\varepsilon_f\partial_{\varepsilon_f}J_f - J_f\right)\frac{J_f}{\varepsilon_f^3} - \left(\varepsilon_f\partial_{\varepsilon_f}H_f^i - H_f^i\right)\frac{H_f^i}{\varepsilon_f^3} + (\varepsilon_f\partial_{\varepsilon_f}K_f^{ij} - K_f^{ij})\frac{K_f^{ij}}{\varepsilon_f^3} - (\varepsilon_f\partial_{\varepsilon_f}Q_f^{ijk} - Q_f^{ijk})\frac{Q_f^{ijk}}{\varepsilon_f^3}\right]\right\} \quad (42)$$

and

$$\partial_{t_f}H_f^i + \partial_{j_f}K_f^{ij} = -n_{e,f}\sigma_T\left\{H_f^i + \frac{2}{5}\left[\left(\frac{T_e - 3\varepsilon_f}{m_e}\right) + \varepsilon_f\partial_{\varepsilon_f}\left(\frac{\varepsilon_f - 4T_e}{m_e}\right) + \frac{T_e}{m_e}\varepsilon_f\partial_{\varepsilon_f}^2\varepsilon_f\right]H_f^i\right.$$

$$\left. + \frac{3}{4}\left(\frac{\varepsilon_f}{m_e}\right)\left[\left(\varepsilon_f\partial_{\varepsilon_f}J_f - J_f\right)\frac{H_f^i}{\varepsilon_f^3} - \left(\varepsilon_f\partial_{\varepsilon_f}H_f^j - H_f^j\right)\frac{K_f^{ij}}{\varepsilon_f^3} + (\varepsilon_f\partial_{\varepsilon_f}K_f^{jk} - K_f^{jk})\frac{Q_f^{ijk}}{\varepsilon_f^3} - (\varepsilon_f\partial_{\varepsilon_f}Q_f^{jkl} - Q_f^{jkl})\frac{R_f^{ijkl}}{\varepsilon_f^3}\right]\right\}, \quad (43)$$

where the coefficients $\mathscr{L}_1-\mathscr{L}_6$, which are given in Appendix A, are to be evaluated at $V = 0$.

The left sides of the above equations can also be written in terms of the system-frame Eulerian coordinates, as in equations (95.9), (95.11), and (95.12) of Mihalas & Mihalas (1984). The resulting equations, which are often called *mixed-frame* equations, are correct to all orders in V and can therefore be used in situations in which the bulk velocity is relativistic. The velocity-dependent terms that appear in the mixed-frame equations arise from the Lorentz transformation of the left sides of equations (41)–(43), whereas the velocity-dependent terms that appear in the system-frame equations arise from the Lorentz transformation of the scattering integral on the right sides of these equations.

The right side of the radiative transfer equation (41) can also be used in the formalism developed by Thorne (1981) for solving Comptonization problems in general relativity (cf. eqs. [6.10] and [6.13] of Thorne 1981).

5. DISCUSSION

In the previous section we derived the radiative transfer equation in both the system and fluid frames, taking into account absorption and emission, as well as spontaneous and induced Compton scattering. In this section we first show that our equation reduces to the Kompaneets equation in the appropriate limits, and we call attention to several errors and misunderstandings in the literature. Next we discuss the moment equations for an anisotropic radiation field in a static medium and show that the radiation force, and hence the critical flux and luminosity, generally depend both on the photon energy spectrum and on the electron temperature. We then consider the equation for the zeroth moment of the specific intensity for moving media and show that if the radiation field is isotropic, the terms in the transfer equation that are second order in the electron bulk velocity produce a systematic increase in the energy of the photons that is completely analogous to the systematic increase in the energy of the photons produced by the electron thermal motions. We also show that Comptonization by electron bulk motion occurs whether or not the radiation field is isotropic or the bulk flow converges, and we give estimates for the timescales on which the photon energy distribution changes because of systematic downscattering and upscattering caused by the electron thermal and bulk motions. We derive a new, more general condition for determining when Comptonization by the electron bulk motion is more important than Comptonization by the electron thermal motions. We conclude by indicating how the transfer equations we have derived can be solved using the method of variable Eddington factors.

5.1. The Kompaneets Equation

When the radiation field is isotropic and there is no bulk velocity and no absorption or emission, equation (34) reduces to the Kompaneets (1957) equation,[1]

$$\partial_{t_C} J = \frac{\varepsilon}{m_e} \partial_\varepsilon(\varepsilon J) - \frac{4 T_e}{m_e} \varepsilon \partial_\varepsilon J + \frac{T_e}{m_e} \varepsilon \partial_\varepsilon^2(\varepsilon J) + \frac{\varepsilon}{m_e} (\varepsilon \partial_\varepsilon J - J) \frac{J}{\varepsilon^3} , \tag{44}$$

where on the left side we have introduced the differential Compton time $dt_C \equiv n_e \sigma_T \, dt$. The first two terms on the right side of equation (44) describe the effects of systematic downscattering and upscattering of the photons by electrons. The third term describes the diffusion in energy produced by the thermal motion of the electrons. The last term describes the effect of induced Compton scattering.

As noted by Kompaneets (1957), *any* Bose-Einstein distribution is a stationary solution of equation (44). Induced Compton scattering cannot change the chemical potential, because it does not change the number of photons. Hence, the statement by Pomraning (1973, p. 193; see also Rybicki & Lightman 1979, p. 209) that the Planck spectrum (the particular Bose-Einstein distribution with zero chemical potential) is the only stationary solution of the Kompaneets equation if induced scattering is included is not correct.

Integrating equation (44) over energy shows that if the electron temperature is equal to the Compton temperature

$$T_C \equiv \frac{\langle \varepsilon^2 \rangle - \langle n \varepsilon^2 \rangle}{4 \langle \varepsilon \rangle} , \tag{45}$$

the photon energy density remains constant although the photon spectrum may evolve with time; here the average is over photon energy, using the photon-number density $N(\varepsilon) \equiv (J/\varepsilon)$ as the weighting function. Note that for a given photon spectrum, induced Compton scattering *always decreases* the Compton temperature.

5.2. Implications of the Moment Equations for Static Media

As we mentioned in § 1, the Kompaneets equation (44) is not strictly valid for astrophysical systems, since it requires the radiation field to be isotropic, and hence that no radiation leave the system. If the radiation field is anisotropic, and there is no bulk motion, the system of equations

$$\partial_t J + \partial_i H^i = \eta - \chi J + n_e \sigma_T \left[\varepsilon \partial_\varepsilon \left(\frac{\varepsilon - 4 T_e}{m_e} \right) + \frac{T_e}{m_e} \varepsilon \partial_\varepsilon^2 \varepsilon \right] J \tag{46}$$

and

$$\partial_t H^i + \partial_j K^{ij} = -n_e \sigma_T H^i - \chi H^i - \frac{2}{5} n_e \sigma_T \left[\left(\frac{T_e - 3\varepsilon}{m_e} \right) + \varepsilon \partial_\varepsilon \left(\frac{\varepsilon - 4 T_e}{m_e} \right) + \frac{T_e}{m_e} \varepsilon \partial_\varepsilon^2 \varepsilon \right] H^i \tag{47}$$

must be solved simultaneously together with closure relations for K^{ij} and J, which are usually introduced as variable Eddington factors (see Mihalas 1978, p. 157). For simplicity we have neglected the effects of induced Compton scattering in equations (46) and (47). The term $(2 n_e \sigma_T/5)[(T_e - 3\varepsilon)/m_e]H^i$ on the right side of equation (47) does not appear in the first moment of the transfer equation derived by Chan & Jones (1975), Payne (1980), and Madej (1989, 1991), because these authors did not use the appropriate relativistic scattering cross section. All the terms in the square brackets on the right side of equation (47) were neglected by Blandford & Payne (1981a), but this is generally unsafe.

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[1] In their derivations of the Kompaneets equation, Rybicki & Lightman (1979, p. 213) and Katz (1987, p. 100) did not take the different phase spaces of the incident and scattered photons into account in the collision integral. However, following Kompaneets, they evaluated the integral by using photon conservation and the thermodynamic equilibrium photon distribution, rather than by performing the integration directly, and they thereby obtained the correct result despite this error. These authors also used the Thomson approximation to the Klein-Nishina cross section. In general this introduces an error of the same size as the systematic downscattering term (see below), but this error vanishes if the photon distribution is isotropic.

In a static medium and in the absence of absorption, emission, and induced scattering, F^i, the radiation force per unit volume on the electrons (see also Lamb & Miller 1995) can be obtained by integrating equation (47) over photon energy, which gives

$$F^i = \frac{4\pi}{c}\, n_e\, \sigma_T \left(1 - \frac{8}{5}\frac{\langle \varepsilon \rangle_{H^i}}{m_e} + 2\frac{T_e}{m_e}\right) \int_0^\infty H^i\, d\varepsilon \, , \tag{48}$$

where $\langle \varepsilon \rangle_{H^i}$ is the average photon energy, using H^i as the weighting function. Blandford & Payne (1981a) neglected the terms in the square brackets on the right side of equation (47), and, therefore, the radiation force given by their equation is incorrect to first order in ε/m_e and T_e/m_e. Equation (48) shows that the volume radiation force on the electrons produced by Compton scattering of the photons is different from that obtained in the Thomson limit and that it depends both on the photon spectrum and on the electron temperature (see also Fukue et al. 1985). Hence, to first order in ε/m_e and T_e/m_e, the energy-integrated critical radial radiation flux $\mathscr{F}^{\mathrm{crit}}$ that produces a radially outward radiation force that exactly balances the inward gravitational force of a massive object depends on the photon spectrum and on the electron temperature. For completely ionized hydrogen gas, the critical radiation flux at radius r is given implicitly by the equation

$$\mathscr{F}^{\mathrm{crit}} = \frac{cm_p\, GM}{r^2 \sigma_T}\left(1 - \frac{8}{5}\frac{\langle \varepsilon \rangle_{\mathscr{F}_\varepsilon^{\mathrm{crit}}}}{m_e} + 2\frac{T_e}{m_e}\right)^{-1} \, , \tag{49}$$

where m_p is the proton mass, M is the mass of the object, and $\langle \varepsilon \rangle_{\mathscr{F}_\varepsilon^{\mathrm{crit}}}$ is the average photon energy, using $\mathscr{F}_\varepsilon^{\mathrm{crit}}$ as the weighting function. For example, in the spectral formation region of many neutron star low-mass X-ray binaries, $\langle \varepsilon \rangle_{\mathscr{F}_\varepsilon^{\mathrm{crit}}}$ is ~ 1 keV, and T_e can be ~ 25 keV (Miller & Lamb 1992; Psaltis et al. 1995), in which case the critical flux is $\sim 10\%$ smaller than the usual Eddington critical flux computed by assuming Thomson scattering. Note also that to this order, the critical luminosity $L_{\mathrm{crit}} \equiv 4\pi r^2 \mathscr{F}^{\mathrm{crit}}$ generally depends on radius, because both $\langle \varepsilon \rangle_{\mathscr{F}_\varepsilon^{\mathrm{crit}}}$ and T_e generally depend on radius.

5.3. Implications of the Moment Equations for Moving Media

Consider now a medium in which the electron bulk velocity is not zero. Suppose first, for simplicity, that the radiation field is isotropic in the *system* frame. To make this situation concrete, consider a thought experiment in which electrons are moving with uniform and constant bulk velocity V through a box with sides of length L. The electrons are assumed to be able to pass through the walls of the box, whereas photons are confined inside the box but have a mean free path to scattering by electrons that is much larger than L. Under these conditions, scattering of photons by the walls of the box, which is much more frequent than scattering of photons by electrons, keeps the photon distribution nearly isotropic.

In this situation the equation for the zeroth moment of the specific intensity reduces to

$$\partial_{t_C} J = \frac{\varepsilon}{m_e}\, \partial_\varepsilon(\varepsilon J) + \left(\frac{T_e}{m_e} + \frac{V^2}{3}\right)(-4\varepsilon \partial_\varepsilon + \varepsilon \partial_\varepsilon^2\, \varepsilon)J \, , \tag{50}$$

where, for simplicity, we have neglected absorption, emission, and induced scattering. Equation (50) shows that when the radiation field is isotropic in the system frame, Comptonization by the bulk motion of the electrons is described entirely by terms that are second order in V; all terms that are first order in V vanish identically.

Suppose now that (i) photons with energies $\varepsilon \ll T_e + \frac{1}{3}m_e V^2$ are injected into the box with an isotropic momentum distribution, (ii) the photons are allowed to remain in the box for a distribution of residence times that decreases exponentially with the residence time, and (iii)

$$y_b \equiv 4\left(\frac{T_e}{m_e} + \frac{1}{3}V^2\right)\bar{t}_{\mathrm{res}} \lesssim 1 \, , \tag{51}$$

where $\bar{t}_{\mathrm{res}}$ is the mean residence time measured in units of the Compton time $(1/n_e \sigma_T)$. The spectrum is then a power-law with a high-energy cutoff (see Sunyaev & Titarchuk 1980). At high and low energies, the spectrum is

$$J = \begin{cases} \varepsilon^{3+\alpha} \, , & \varepsilon \ll T_e + \frac{1}{3}m_e V^2 \, , \\ \varepsilon^3 \exp\left[-\varepsilon/(T_e + \frac{1}{3}V^2)\right] \, , & \varepsilon \gg T_e + \frac{1}{3}m_e V^2 \, , \end{cases} \tag{52}$$

where

$$\alpha \equiv -\frac{3}{2} - \sqrt{\frac{9}{4} + \frac{4}{y_b}} \, , \tag{53}$$

and the factor ε^3 arises because we are considering the energy density rather than the photon occupation number. This solution is a simple generalization of the solution to the Kompaneets equation obtained by Shapiro, Lightman, & Eardley (1976; see also Rybicki & Lightman 1979, chap. 7). It shows that *the terms in the transfer equation that are second order in the electron bulk velocity produce a systematic increase in the energy of the photons that is completely analogous to the systematic increase in the energy of the photons produced by the electron thermal motions*, as argued in § 2.2.

When the photon momentum distribution is not perfectly isotropic, the characteristic timescales on which the photon distribution changes because of systematic downscattering and Comptonization by the electron thermal and bulk motions are

(see eq. [34])

$$t_{\text{down}}^{-1} \sim n_e \sigma_T \left(\frac{\varepsilon}{m_e}\right),$$ (54)

$$t_{\text{th}}^{-1} \sim n_e \sigma_T \left(\frac{4T_e}{m_e}\right),$$ (55)

$$t_V^{-1} \sim n_e \sigma_T V \cdot \left(\frac{H}{J}\right),$$ (56)

$$t_{V2}^{-1} \sim n_e \sigma_T \left(\frac{4V^2}{3}\right),$$ (57)

where the last two timescales have been estimated from the terms that are first and second order in V.

Comparison of rates (55) and (57) shows that Comptonization by the electron bulk motion is more important than Comptonization by the electron thermal motion if

$$V^2 > \frac{3T_e}{m_e}.$$ (58)

In fact, as explained in § 2.2, Comptonization by the bulk motion occurs whether or not the radiation field is isotropic or the bulk motion converges (see also eq. [50]), contrary to the impression given by Blandford & Payne (1981a). Comparison of rates (56) and (57) shows that *if J is sufficiently large compared to H^i, the effects of bulk Comptonization described by the terms that are second order in V are dominant compared to the effects described by the terms that are first order in V.*

In estimating the characteristic timescale for bulk Comptonization, Blandford & Payne (1981a; see also Blandford & Payne 1981b and Payne & Blandford 1981) used only one of the several terms in their equation that are first order in V, neglecting other terms that generally also produce systematic upscattering or downscattering of photons. As a result, the characteristic timescale that they obtained for bulk Comptonization is proportional to $(\mathbf{V} \cdot \mathbf{V})/3$, which they assumed to be proportional to $n_e \sigma_T V^2/3$. This assumption is not generally valid. When it is, their expression suggests that bulk Comptonization is less important than thermal Comptonization if $V^2 < 12T_e/m_e$. However, comparison of rates (55)–(57) shows that the bulk Comptonization terms that are second order in V are already as important as the thermal Comptonization terms when $V^2 \sim 3T_e/m_e$, and hence, that these terms can be more important than the thermal terms even if $V^2 < 12T_e/m_e$.

5.4. A Method of Solving the Transfer Equation for Moving Media

In more realistic models, the radiation field is not isotropic in the system frame in the presence of bulk motion (see, e.g., Miller & Lamb 1993, 1996), because electron scattering tends to isotropize the photon distribution in the *fluid* frame. In this case, the full radiative transfer equation must be solved. This can be done by using the method of variable Eddington factors (Mihalas 1980; see also Mihalas 1978, p. 201), in which the radiative transfer equation and its zeroth and first moments are solved iteratively. In this approach, the second and higher moments of the specific intensity are related to the zeroth and first moments via variable Eddington factors. The zeroth and first moments of the radiative transfer equation are then solved by using initial guesses for the variable Eddington factors, and the source function is computed from the calculated moments of the specific intensity. The radiative transfer equation is then solved, the Eddington factors are updated, and the whole procedure is repeated until convergence is achieved.

A detailed study of the solutions of the equations derived here for realistic models of astrophysical systems will be reported elsewhere.

APPENDIX A

PHOTON KINETIC AND RADIATIVE TRANSFER EQUATIONS FOR SCATTERING BY AN ELECTRON FLUID

In this appendix we give the photon kinetic equation and the corresponding radiative transfer equation for scattering by an electron gas with temperature T_e and bulk velocity V.

We start from the photon kinetic equation (27) for scattering by electrons with a given velocity, transform all the quantities to the system frame using equations (18) (19), and then average over the electron velocity distribution. The resulting photon kinetic equation is

$$\frac{1}{n_e \sigma_T} (\partial_t + l^i \partial_i) n(\hat{l}, \varepsilon) = \mathscr{R}_1 n(\hat{l}, \varepsilon) + \frac{3}{4} (\mathscr{R}_2 n + \mathscr{R}_3^i n^i + \mathscr{R}_4^{ij} n^{ij} + \mathscr{R}_5^{ijk} n^{ijk} + \mathscr{R}_6^{ijkl} n^{ijkl})$$

$$+ \frac{3}{2} \left(\frac{\varepsilon}{m_e}\right) n(\hat{l}, \varepsilon)(2 + \varepsilon \partial_\varepsilon)(n - l^i n^i + l^i l^j n^{ij} - l^i l^j l^k n^{ijk}),$$ (A1)

where

$$\mathcal{R}_1 = -1 + 2\frac{\varepsilon}{m_e} + (\hat{l} \cdot V) \,, \tag{A2}$$

$$\mathcal{R}_2 = 1 + 2\frac{\varepsilon}{m_e} + \frac{\varepsilon}{m_e}\varepsilon\partial_\varepsilon + 2\frac{T_e}{m_e} - (\hat{l} \cdot V) + (\hat{l} \cdot V)^2 - V^2 + \varepsilon\partial_\varepsilon\left[4\frac{T_e}{m_e} - (\hat{l} \cdot V) + (\hat{l} \cdot V)^2\right] + \frac{1}{2}\varepsilon^2\partial_\varepsilon^2\left[2\frac{T_e}{m_e} + (\hat{l} \cdot V)^2\right], \tag{A3}$$

$$\mathcal{R}_3^i = -2\frac{\varepsilon}{m_e}l^i - \frac{\varepsilon}{m_e}\varepsilon\partial_\varepsilon l^i - 4\frac{T_e}{m_e}l^i + 2V_i - 2(\hat{l} \cdot V)l^i - 2(\hat{l} \cdot V)^2 l^i + 2V^2 l^i$$
$$+ \varepsilon\partial_\varepsilon\left[-4\frac{T_e}{m_e}l^i + V_i - 4(\hat{l} \cdot V)V_i + 2(\hat{l} \cdot V)^2 l^i\right] + \frac{1}{2}\varepsilon\partial_\varepsilon^2\left[-2\frac{T_e}{m_e}l^i - 2(\hat{l} \cdot V)V_i\right], \tag{A4}$$

$$\mathcal{R}_4^{ij} = l^i l^j + 2\frac{\varepsilon}{m_e}l^i l^j + \frac{\varepsilon}{m_e}\varepsilon\partial_\varepsilon l^i l^j - 6\frac{T_e}{m_e}l^i l^j + (\hat{l} \cdot V)l^i l^j - 2l^i V_j + (\hat{l} \cdot V)^2 l^i l^j - 8(\hat{l} \cdot V)l^i V_j - 3V^2 l^i l^j + 4V_i V_j$$
$$+ \varepsilon\partial_\varepsilon\left[4\frac{T_e}{m_e}l^i l^j - (\hat{l} \cdot V)l^i l^j + 3V_i V_j - (\hat{l} \cdot V)^2 l^i l^j\right] + \frac{1}{2}\varepsilon\partial_\varepsilon^2\left[2\frac{T_e}{m_e}l^i l^j + V_i V_j + (\hat{l} \cdot V)^2 l^i l^j\right], \tag{A5}$$

$$\mathcal{R}_5^{ijk} = -2\frac{\varepsilon}{m_e}l^i l^j l^k - \frac{\varepsilon}{m_e}\varepsilon\partial_\varepsilon l^i l^j l^k + 4\frac{T_e}{m_e}l^i l^j l^k + 4l^i l^j V_k - 8l^i V_j V_k + 4(\hat{l} \cdot V)l^i l^j V_k$$
$$+ \varepsilon\partial_\varepsilon\left[-4\frac{T_e}{m_e}l^i l^j l^k + l^i l^j V_k - 4(\hat{l} \cdot V)l^i l^j V_k - 2l^i V_j V_k\right] + \frac{1}{2}\varepsilon^2\partial_\varepsilon^2\left[-2\frac{T_e}{m_e}l^i l^j l^k - 2(\hat{l} \cdot V)l^i l^j V_k\right], \tag{A6}$$

$$\mathcal{R}_6^{ijkl} = 10l^i l^j V_k V_l + 5\varepsilon\partial_\varepsilon l^i l^j V_k V_l + \frac{1}{2}\varepsilon^2\partial_\varepsilon^2 l^i l^j V_k V_l. \tag{A7}$$

The corresponding radiative transfer equation is

$$\frac{1}{n_e\sigma_T}(\partial_t + l^i\partial_i)I(\hat{l}, \varepsilon) = \mathcal{L}_1 I(\hat{l}, \varepsilon) + \frac{3}{4}(\mathcal{L}_2 J + \mathcal{L}_3^i H^i + \mathcal{L}_4^{ij} K^{ij} + \mathcal{L}_5^{ijk} Q^{ijk} + \mathcal{L}_6^{ijkl} R^{ijkl})$$
$$+ \frac{3}{4\varepsilon^3}\left(\frac{\varepsilon}{m_e}\right)I(\hat{l}, \varepsilon)(\varepsilon\partial_\varepsilon - 1)(J - l^i H^i + l^i l^j K^{ij} - l^i l^j l^k Q^{ijk}) \,, \tag{A8}$$

where

$$\mathcal{L}_1 = -1 + 2\frac{\varepsilon}{m_e} + (\hat{l} \cdot V) \,, \tag{A9}$$

$$\mathcal{L}_2 = 1 - \frac{\varepsilon}{m_e} + \frac{\varepsilon}{m_e}\varepsilon\partial_\varepsilon + 2\frac{T_e}{m_e} + 2(\hat{l} \cdot V) + 4(\hat{l} \cdot V)^2 - V^2$$
$$+ \varepsilon\partial_\varepsilon\left[-2\frac{T_e}{m_e} - (\hat{l} \cdot V) - 2(\hat{l} \cdot V)^2\right] + \frac{1}{2}\varepsilon^2\partial_\varepsilon^2\left[2\frac{T_e}{m_e} + (\hat{l} \cdot V)^2\right], \tag{A10}$$

$$\mathcal{L}_3^i = \frac{\varepsilon}{m_e}l^i - \frac{\varepsilon}{m_e}\varepsilon\partial_\varepsilon l^i - 4\frac{T_e}{m_e}l^i - V_i - 2(\hat{l} \cdot V)l^i - 8(\hat{l} \cdot V)^2 l^i + 2V^2 l^i$$
$$+ \varepsilon\partial_\varepsilon\left[2\frac{T_e}{m_e}l^i + V_i + 2(\hat{l} \cdot V)V_i + 2(\hat{l} \cdot V)^2 l^i\right] + \frac{1}{2}\varepsilon\partial_\varepsilon^2\left[-2\frac{T_e}{m_e}l^i - 2(\hat{l} \cdot V)V_i\right], \tag{A11}$$

$$\mathcal{L}_4^{ij} = l^i l^j - \frac{\varepsilon}{m_e}l^i l^j + \frac{\varepsilon}{m_e}\varepsilon\partial_\varepsilon l^i l^j - 6\frac{T_e}{m_e}l^i l^j + 4(\hat{l} \cdot V)l^i l^j - 2l^i V_j + 10(\hat{l} \cdot V)^2 l^i l^j - 8(\hat{l} \cdot V)l^i V_j - 3V^2 l^i l^j + V_i V_j$$
$$+ \varepsilon\partial_\varepsilon\left[-2\frac{T_e}{m_e}l^i l^j - (\hat{l} \cdot V)l^i l^j - 4(\hat{l} \cdot V)^2 l^i l^j\right] + \frac{1}{2}\varepsilon\partial_\varepsilon^2\left[2\frac{T_e}{m_e}l^i l^j + V_i V_j + (\hat{l} \cdot V)^2 l^i l^j\right], \tag{A12}$$

$$\mathcal{L}_5^{ijk} = \frac{\varepsilon}{m_e}l^i l^j l^k - \frac{\varepsilon}{m_e}\varepsilon\partial_\varepsilon l^i l^j l^k + 4\frac{T_e}{m_e}l^i l^j l^k + l^i l^j V_k - 2l^i V_j V_k + 4(\hat{l} \cdot V)l^i l^j V_k$$
$$+ \varepsilon\partial_\varepsilon\left[2\frac{T_e}{m_e}l^i l^j l^k + l^i l^j V_k + 2(\hat{l} \cdot V)l^i l^j V_k - 2l^i V_j V_k\right] + \frac{1}{2}\varepsilon^2\partial_\varepsilon^2\left[-2\frac{T_e}{m_e}l^i l^j l^k - 2(\hat{l} \cdot V)l^i l^j V_k\right], \tag{A13}$$

$$\mathcal{L}_6^{ijkl} = l^i l^j V_k V_l + 2\varepsilon\partial_\varepsilon l^i l^j V_k V_l + \frac{1}{2}\varepsilon^2\partial_\varepsilon^2 l^i l^j V_k V_l \,. \tag{A14}$$

APPENDIX B

EMISSION AND ABSORPTION PROCESSES IN THE SYSTEM FRAME

In the absence of scattering, the transfer equation in the system frame becomes simply (see, e.g., Mihalas & Mihalas 1984, p. 422)

$$(\partial_t + l^i \partial_i) I(\hat{l}, \varepsilon) = \left(\frac{\varepsilon}{\varepsilon_f}\right)^2 \eta(\varepsilon_f) - \left(\frac{\varepsilon_f}{\varepsilon}\right) \chi(\varepsilon_f) I(\hat{l}, \varepsilon) , \tag{B1}$$

where again the subscript f refers to quantities in the fluid frame, and we have, for simplicity, assumed that the emission coefficient η and the absorption coefficient χ are isotropic in this frame. In writing equation (B1) we have also used the fact that the absorption and emission coefficients are defined in the fluid frame and that (η/ε^2) and $(\varepsilon \chi)$ are Lorentz invariants.

We can now expand $\eta(\varepsilon_f)$ to second order in V as

$$\eta(\varepsilon_f) = \eta[\varepsilon + (\varepsilon_f - \varepsilon)] \simeq \eta_\varepsilon + (\varepsilon_f - \varepsilon)\partial_\varepsilon \eta_\varepsilon + \tfrac{1}{2}(\varepsilon_f - \varepsilon)^2 \partial_\varepsilon^2 \eta_\varepsilon \simeq \eta_\varepsilon + (\tfrac{1}{2}V^2 - \hat{l} \cdot V)\varepsilon \partial_\varepsilon \eta_\varepsilon + (\hat{l} \cdot V)^2 \varepsilon^2 \partial_\varepsilon^2 \eta_\varepsilon , \tag{B2}$$

where $\eta_\varepsilon = \eta(\varepsilon)$. We also expand $\chi(\varepsilon_f)$ in a similar way. The transfer equation to second order in V then becomes

$$(\partial_t + l^i \partial_i) I(\hat{l}, \varepsilon) = \{[1 + 2(\hat{l} \cdot V) + 3(\hat{l} \cdot V)^2 - V^2] + [-(\hat{l} \cdot V) - 2(\hat{l} \cdot V)^2 + \tfrac{1}{2}V^2]\varepsilon \partial_\varepsilon + \tfrac{1}{2}(\hat{l} \cdot V)^2 \varepsilon^2 \partial_\varepsilon^2\}\eta_\varepsilon$$
$$- \{[1 - (\hat{l} \cdot V) + \tfrac{1}{2}V^2] + [-(\hat{l} \cdot V) + (\hat{l} \cdot V)^2 + \tfrac{1}{2}V^2]\varepsilon \partial_\varepsilon + \tfrac{1}{2}(\hat{l} \cdot V)^2 \varepsilon \partial_\varepsilon\}\chi_\varepsilon I(\hat{l}, \varepsilon) . \tag{B3}$$

We can then obtain the zeroth and first moments of the transfer equation,

$$\partial_t J + \partial_i H^i = \eta_\varepsilon - \chi_\varepsilon J + V^i H^i[\chi_\varepsilon + \varepsilon \partial_\varepsilon \chi_\varepsilon] - \tfrac{1}{6}V^2 \varepsilon \partial_\varepsilon \eta_\varepsilon + \tfrac{1}{6}V^2 \varepsilon^2 \partial_\varepsilon^2 \eta_\varepsilon$$
$$- \tfrac{1}{2}\chi_\varepsilon J V^2 - (\tfrac{1}{2} J V^2 + K^{ij} V_i V_j)\varepsilon \partial_\varepsilon \chi_\varepsilon - \tfrac{1}{2}K^{ij} V_i V_j \varepsilon^2 \partial_\varepsilon^2 \chi_\varepsilon \tag{B4}$$

and

$$\partial_t H^i + \partial_j K^{ij} = -\chi_\varepsilon H^i + \tfrac{1}{3}V^i(2\eta_\varepsilon - \varepsilon \partial_\varepsilon \eta_\varepsilon) + K^{ij} V_j (\chi_\varepsilon + \varepsilon \partial_\varepsilon \chi_\varepsilon) - \tfrac{1}{2}\chi_\varepsilon H^i V^2 - (\tfrac{1}{2}H^i V^2 + Q^{ijk} V_j V_k)\varepsilon \partial_\varepsilon \chi_\varepsilon - \tfrac{1}{2}Q^{ijk} V_j V_k \varepsilon^2 \partial_\varepsilon^2 \chi_\varepsilon . \tag{B5}$$

Blandford & Payne (1981a) added a photon source term to the right side of their transfer equation without adding any corresponding absorption term. This is fundamentally inconsistent with thermodynamics: if absorption is not included, the photon source can never come into thermodynamic equilibrium with the radiation field. It is equivalent to including the terms on the right sides of equations (B3)–(B5) that involve the emission coefficient η_ε but neglecting all the terms that involve the absorption coefficient χ_ε.

Neglecting the absorption terms but not the emission term is a valid *approximation* only if the specific intensity of the radiation field is negligible compared to the source function $\mathscr{S}_\varepsilon = \eta_\varepsilon/\chi_\varepsilon$, at all photon energies, i.e., only if self-absorption is never important at any energy, anywhere in the system. This is rarely the case in astrophysical systems. For example, it is not the case in the Comptonizing regions around accreting neutron star and black hole X-ray sources.

Neglecting the absorption terms in the transfer equation leads to equations that have a different mathematical character from equations (B3)–(B5), because in this case the right sides of the transfer and moment equations do not depend explicitly on the radiation field.

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